Present-Bias and the Value of Sophistication*

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Abstract

This paper develops a dynamic wealth management model for risk-averse investors displaying present-bias in the form of hyperbolic discounting. The investor chooses an optimal consumption policy and allocates her funds between a risk-free asset, a traded liquid asset, and a non-traded illiquid asset. We characterize these policies for both sophisticated and naive present-biased investors. There are three results. First, sophisticated investors over-consume more than their naive counterparts if and only if their coefficient of relative risk-aversion is smaller than one. As a result, sophistication is welfare reducing (increasing) when risk-aversion is low (high). Second, increasing asset illiquidity always benefits the sophisticated investor more than the naive investor. Thus, the welfare gap between sophisticated and naive investors is increasing in the proxy for asset illiquidity. Finally, present-biased investors accumulate a larger share of their wealth in the non-traded illiquid asset than in the traded risky stock compared to the neoclassical exponential discounter investor. As a consequence, from the perspective of present-biased investors, the equity premium puzzle (Mehra and Prescott, 1985) and the private equity puzzle (Moskowitz and Vissing-Jørgensen, 2002) are two sides of the same coin.

Keywords: Present-Bias, Behavioral Finance, Private Equity Puzzle, Commitment Devices.

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1 Introduction

Aversion towards risk and a bias towards the present are two well established features influencing the decision-making process of economic agents. Risk-aversion, on the one hand, has been a central element of economic theory since its onset. Its formal introduction to the literature dating back to at least De Finetti (1952) and Pratt (1964). Present-bias, on the other hand, has been documented by a large body of experimental and field evidence. Strotz (1955) provided its first formal treatment (subsequently expanded by Phelps and Pollak, 1968; Pollak, 1968), but it was only after Laibson (1997) that the notion of present-bias really took hold in the economics literature. Moreover, the distinction of whether the agent is sophisticated enough to account for the presence of this bias when planning for the future has had important implications in the finance (e.g., Angeletos *et al.* (2001)), industrial organization (e.g., DellaVigna and Malmendier (2004); O'Donoghue and Rabin (1999)), and contract theory literature (e.g., Gottlieb and Zhang (2020)).

In this paper, we explore the interaction between risk-aversion and present-bias in the consumption and investment decisions of investors, and assess the welfare implication of behavioral sophistication within this setting. We show that being aware of one's present-bias (i.e., sophistication) is welfare enhancing when the investor's coefficient of relative risk-aversion is larger than one. By contrast, being unaware about one's present-bias (i.e., naiveté) is welfare enhancing when the coefficient of relative risk-aversion is smaller than one. Importantly, our result is robust to the inclusion of liquid and illiquid assets, borrowing constraints, as well as the presence of idiosyncratic and systematic risks.

Before delving into the intuition for this result, we provide more details about our model. We model the investor as a sequence of selves. The current self controls current consumption and investment decisions, but derives utility from the entire stream of consumption chosen by her future selves. We lever on the game theoretic literature to interpret our model as a game played between the investor's current self and her future selves. We characterize the solution of this game using Markov Perfect Equilibrium (MPE) as the solution concept. Furthermore, we allow for investors to have heterogeneous degrees of sophistication regarding their present bias. We focus our analysis in the two extreme cases: complete sophistication and full naiveté. Sophistication implies that the current self is fully aware that her future selves will display the same amount of present-bias as her current self, and thus makes decisions anticipating this bias. By contrast, the naive thinks that only her current self displays present bias, and (wrongly) anticipates that her future selves will not display any amount of present bias (i.e., that her future selves will be exponential discounters).

For tractability reasons we assume time to be continuous. We adopt the continuous-time hyperbolic discounting framework developed by Luttmer and Mariotti (2003) and extended by Harris and Laibson (2013). We model two state variables. The investor's liquid wealth, invested in the risk-free bond and the traded (liquid) risky stock, and the private firm's size (illiquid private equity). The stock follows a standard geometric brownian motion with constant coefficients a la Merton (1973). The investor can costlessly adjust her share of wealth exposed to the stock and the risk-free asset at each point in time. By contrast, the size of the private firm follows a controlled geometric brownian motion with neoclassical adjustment costs (e.g., Hayashi, 1982). We interpret the magnitude of the adjustment costs as a proxy for asset illiquidity. Higher adjustment costs means it is costly for the investor to liquidate a large fraction of her private firm over a short period of time. Thus, investing in private equity provides the investor with a commitment technology to prevent her future selves from splurging.

Our main result states that the naive present-biased investor over-consumes more than her sophisticated counterpart if and only if the coefficient of relative risk-aversion is larger than one. As result, according to the long-run welfare criterion put forth by O'Donoghue and Rabin (1999), sophistication is welfare reducing (increasing) when relative risk aversion is below (above) one. Because sophistication makes the investor cognizant that her future selves will also display present-bias, our results emerge from the interplay of two effects. On the one hand, the sophisticated investor anticipates that her future selves will splurge. Thus, saving becomes less appealing, and the investor increases over-consumption. We refer to this force as the *splurging effect*. On the other hand, the sophisticated investor correctly anticipates that splurging will lead her future selves to deplete her wealth and push her into states where the marginal utility of consumption is very high. Preventing this undesirable outcome makes saving (by the current-self) more appealing, thereby leading to less over-consumption. We refer to this mechanisms as the *precautionary effect*. It turns out that when risk aversion is high (i.e., relative risk-aversion above one) the precautionary effect dominates and the sophisticated investor over-consumes less than the naive investor. When risk-aversion is low (i.e., relative risk-aversion below one) the splurging effect dominates and the sophisticated investor over-consumes more than the naive. Finally, as expected, when risk aversion is exactly equal to one (logarithmic preferences), these two effects exactly cancel each other out and the consumption policies of both types of investors coincide.[1]

[1]Pollak (1968) already pointed out the coincidence of behavior for naive and sophisticated for the case of a

There are three important additional results. First, higher asset illiquidity always benefits the sophisticated investor more than his naive counterpart. As a result, the gap between the welfare of the sophisticated investor and the naive investor is increasing in the proxy for illiquidity of the private equity investment. Because the sophisticated investor is aware of her future present bias, higher asset illiquidity endows her with a more powerful commitment device to constraint her future selves from splurging. The naive investor, by contrast, does not anticipate being present-biased in the future. Thus, she does not benefit from the presence of commitment devices.[2] A potentially fruitful policy intervention would involve making investors more self-aware of their future present-bias, while simultaneously providing them with commitment devices to mitigate the adverse effect of such bias. This also implies that incentivizing entrepreneurship can have a better than expected return, if it improves the commitment abilities of sophisticated investors.

Second, the value of the illiquid private equity as a commitment device is decreasing in its volatility. This is quite intuitive: the value of a commitment device lies precisely in allowing the current self to influence the consumption of her future selves, and an increase in volatility decreases her ability to set future consumption. Therefore, for a given fixed return in the private equity, an increase in its volatility reduces its value as a commitment device for risk-averse agents.

Finally, we explore the fraction of wealth accumulated by the investor in the traded versus the non-traded assets. In line with our main result, we show that the sophisticated investor accumulates more (less) wealth in the traded asset than the naive investor when her risk-aversion is high (low). Moreover, present-biased investors always accumulate a larger fraction of their wealth in the private firm than in the risky stock, compared to exponential investors. Therefore, in general equilibrium, an economy populated with present-biased investors would feature higher equity premia for traded (liquid) assets and lower expected returns for non-traded (illiquid) assets. Consequently, from the perspective of a model of present-biased investors, the equity premium puzzle (Mehra and Prescott, 1985)) and the private equity puzzle (Moskowitz and Vissing-Jørgensen, 2002) are deeply intertwined, as two sides of the same coin.

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logarithmic utility. Below we discuss the literature that has focused on the issue of naive vs. sophisticated, and especially their interaction with respect to the coefficient of relative risk-aversion.

[2]It is well known that commitment devices help sophisticated hyperbolic discounters alleviate their dynamic inconsistency problems by "solidifying" part of their future behavior in the present moment. For example, an agent that deposits her money in an illiquid bank account bearing no interest (as in "Christmas clubs"), that would only allow to recover her money back in six months time, does so to avoid having her impatient future selves consume excessively (Laibson, 1997; Giné *et al.*, 2010; Duflo *et al.*, 2011). Beshears *et al.* (2020) also show that for naive investors, providing some amount of illiquidity can be optimal from a social welfare perspective. Importantly, our model implies that efforts to provide present-biased investors with commitment devices will be more effective for investors that display sophistication regarding their present-bias.

Literature review

One of the oldest results in the analysis of hyperbolic discounting (and behavioral economics more generally), is the fact that the behavior of naive and sophisticated agents coincides for a logarithmic utility function (Pollak, 1968). This result has been extended to the Merton model (Marin-Solano and Navas, 2010). Crucially for us, the logarithmic function corresponds to a CRRA with $\gamma = 1$, and one of our contributions is to show that this value for γ actually constitutes a threshold at which the behavior of naive and sophisticated agents reverses. Our paper is therefore connected to a literature that has found that behavior switches at the threshold $\gamma = 1$ for agents with CRRA utility functions. Gollier *et al.* (1997) found that agents with higher flexibility invested more in risky assets for $\gamma < 1$, and the reverse holds for $\gamma > 1$ (as we will see in Section 4, this intuition translates to hyperbolic discounting). More relevant for us, Barro (1999) noted (in a deterministic environment) that for CRRA functions the behavior of hyperbolic discounters reverses at the threshold $\gamma = 1$, but in contrast to our results, he focused on the influence of γ on interest rates rather than emphasizing the distinction of naive vs. sophisticated agents.[3]

A recent literature analyzes hyperbolic discounting for CRRA utility functions in the Merton model (Marin-Solano and Navas, 2010; Zou *et al.*, 2014).[4] We contribute to this literature by focusing on the interaction between the coefficient of risk aversion γ and the agent sophistication, and are the first to prove that there is a threshold at $\gamma = 1$ such that magnitude of overconsumption for naive and sophisticated agents reverses as γ crosses the threshold.

Brocas and Carrillo (2004) consider an economy of sophisticated entrepreneurs with hyperbolic discounting, and their focus is on endogenous acquisition of information: they find that some entrepreneurs might prefer to forgo information, in order to "discipline" future selves. DellaVigna and Malmendier (2004) study the contracts that a monopolist offers to hyperbolic agents, and show that the contracts offered include switching costs (as those agents underestimate the probability of incurring those costs). Gottlieb and Zhang (2020) analyze contracts between firms and consumers in the long-run, with one- and two-sided commitment, and show that hyperbolic discounting can actually make some front-loaded contracts more feasible (as present-biased agents discount the future less steeply than exponential discounters). While the last two papers explore how commit-

[3] Others have focused on analyzing CRRA utility functions in hyperbolic discounting only for a region of the parameter space such as $\gamma > 1$ (e.g., Tsoukis *et al.* (2017)).

[4] Ekeland and Pirvu (2008) analyze the problem using a non-exponential form of discounting called pseudo-exponential, and Grenadier and Wang (2007) solve the problem of optimally exercising an option for naive vs. sophisticated agents with linear utility functions.

ment (or lack thereof) is exploited by firms who face present-biased consumers, our focus is on how hyperbolic entrepreneurs can generate commitment by investing (inefficiently high, as compared to an exponential discounter) in their own firm.[5]

It has been previously emphasized (O'Donoghue and Rabin, 1999) that whether an agent is naive or sophisticated about their present bias has important implications for their behavior. As mentioned before, DellaVigna and Malmendier (2004) show that a monopolist would incorporate switching costs to their products to exploit hyperbolic agents but, interestingly, only naive agents would have a reduction in welfare. Within the literature on corporate finance in continuous time with time inconsistent preferences, Grenadier and Wang (2007) apply the IG model developed by Harris and Laibson (2013) to study the exercise of real options under hyperbolic discounting, and show that the timing of the exercise depends on whether the investor is naive or sophisticated and whether the payoff occurs all at once, or over time. Tian (2016) expands on this analysis to study capital structure decisions and shows that hyperbolic discounting causes inefficiencies in the timing of both investment and default. She shows that naive investors choose higher levels of leverage than their sophisticated counterparts. Our focus is on entrepreneurial finance where the investor chooses her firms investment policy as well as her portfolio and consumption decisions. We show that the investor's attitude towards sophistication is critical for her welfare, and show that illiquid private equity disproportionally benefits the sophisticated investors compared to her naive counterpart.[6]

2 Model

2.1 Dynamics and investment opportunity set

An investor has two types of investment opportunities, traded financial assets and private equity from her business. The per period cashflows from the private equity line of business are proportional to firm size X_t. Firm size evolves according to a controlled geometric brownian motion process (GBM):

$$dX_t = X_t((\mu_X + i_t)dt + \sigma_X dB_t^X), \tag{2.1}$$

[5]Developments in the recent literature of contracting with behavioral agents are summarized in Grubb (2015) and Koszegi (2014)

[6]Li *et al.* (2020) analyze the case of a hyperbolic fund manager, and obtain results similar to Grenadier and Wang (2007), in that the manager prefers more leverage than if she had exponential time discounting. Relatedly, Mu *et al.* (2016) characterize the optimal contract in a dynamic principal-agent setting when the agent exhibits hyperbolic discounting.

where B_t^X is a standard brownian motion process under probability measure $\mathbb{P}$, and parameters $\mu_X, \sigma_X > 0$ represent the baseline growth and volatility of profitability, respectively. Firm size can be increased via investment i_t. Investment costs satisfy standard quadratic adjustment costs $g(i) = i + \theta\frac{i^2}{2}$.

The investors can also invest in traded financial assets. In particular, she can invest in a risk-free bond with return r and in a risky stock with cumulative return R_t that evolves according to

$$\frac{dR_t}{R_t} = \mu_S dt + \sigma_S dB_t^S,$$

where B_t^S is a standard brownian motion independent of B_t^X, and $\mu_S, \sigma_S > 0$ represent the expected return and volatility of the risky stock, respectively.

The investor's financial wealth W_t evolves according to

$$dW_t = rW_t dt + \pi_t W_t \left((\mu_S - r)dt + \sigma_S dB_t^S\right) + X_t dt - X_t g(i_t)dt - C_t dt, \tag{2.2}$$

where π_t represents the fraction of wealth invested in the risky stock and C_t the investor's consumption.

We assume the VNM utility function over consumption for the investor is given by CRRA preferences:

$$u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma},$$

where γ represents the relative risk aversion coefficient.

The investor is subject to financial constraints that require her financial wealth to be non-negative at all times. The following definition specifies admissible policies that guarantee the financial constraint is respected.

Definition 1. A policy (C, i, π) is in the admissible set $A(X_0, W_0)$ if X_t satisfies (2.1) with initial condition X_0, wealth W_t satisfies (2.2) with initial condition W_0, and the borrowing constraint of the investor $W_t \geq 0$ is satisfied for all t.

2.2 Present-Bias

We now describe a discount function that models present-bias preferences in continuous time. All periods, present and future, are discounted exponentially with discount factor $0 < \delta < 1$. However, future periods are additionally discounted with uniform weight $0 < \beta \leq 1$. As a result, the present period receives full weight, while future periods are given weight $\beta e^{-\delta t}$.

We model our economic agent (the investor) as a sequence of selves. Call the self born at time $s_0 = 0$ "self 0". The lifetime of self 0 is split into the present, which lasts from s_0 to $s_0 + \tau_0$, and the future, which lasts from $s_0 + \tau_0$ to ∞. The present can be thought of as the interval during which control is exercised by self 0, while the future is the interval during which control is exercised by subsequent selves. The length of the time interval τ_0 is stochastic and exponentially distributed with hazard rate $\lambda \in [0, \infty)$.

Once the present of self 0 ends at $s_0 + \tau_0$, self 1 is born and takes control. The preferences of self 1 are identical to those of self 0, and her present lasts from $s_1 = s_0 + \tau_0$ to $s_1 + \tau_{1.}$ Proceeding in this manner we obtain an infinite sequence of selves $\{0, 1, 2, ...\}$ born respectively at dates $\{s_0, s_1, ...\}$ Each self applies a discount factor $D_n(t)$ to the utility flow at time $s_n + t$, where

$$D_n(t) = \left\{ \begin{array}{c} e^{-\delta t} \ \textit{if}\ t \in [0, \tau_n) \\ \beta e^{-\delta t} \ \textit{if}\ t \in [\tau_n, \infty) \end{array} \right\}.$$

For tractability reasons we will focus in the limiting case when $\lambda \to \infty$, known as the Instantaneous Gratification (IG) Model pioneered by Harris and Laibson (2013). In this case the discount function exhibits a discrete discontinuity at $t = 0_+$ so that

$$D(t) = \left\{ \begin{array}{c} 1 \ \textit{if}\ t = 0 \\ \beta e^{-\delta t} \ \textit{if}\ t \in (0, \infty) \end{array} \right\}.$$

Note that setting $\beta = 1$ nests as a special case the standard exponential discounting used in most economic applications.

Finally, a key consideration in our analysis is whether the agent correctly anticipates the present bias she will have in the future. Following O'Donoghue and Rabin (2001), we suppose that the entrepreneur believes her future selves β will be $\hat{\beta}$. We analyze two cases: the sophisticated case ($\beta = \hat{\beta}$) whereby the investor understands her future selves will behave differently than she would like to, and the naive case ($\hat{\beta} = 1$) whereby the investor ignores the inconsistency in her preferences and (incorrectly) anticipates her future selves will behave as she would like to.

3 Model solution

In this section we first solve the model for the case in which the investor has sophisticated beliefs about the behavior of her future selves. We then solve for the case in which the investor holds naive beliefs. Finally, we show how to compute the ergodic distribution implied by each of these models. The ergodic distribution will allow us to study the implications of present-bias for the steady-state levels of consumption, investment in financial assets, and investment in the private equity.

3.1 Solution under sophisticated beliefs

For the case of sophisticated beliefs, our investor is modeled as a sequence of autonomous selves. Hence, our consumption-investment problem is an intrapersonal game. We will use the notion of Markov perfect equilibrium (MPE) as the solution concept for her game. Moreover, we restrict ourselves to stationary MPE. That is, we focus on equilibria in which all selves use the same strategy.

Denote the consumption, investment, and portfolio strategies of the sophisticated investor by $C^S(X,W), i^S(X,W)$, and $\pi^S(X,W)$, respectively; and the value for the current self by $V(X,W)$. Recall the Instantaneous Gratification (IG) Model specifies that the current self lives only for a vanishingly short time interval, so that

$$V(X,W) = \beta E_t \left[\int_t^\infty e^{-\delta(s-t)} u\left(C(X_s,W_s)\right) ds \right] = \beta F^S(X,W),$$

where $F^S(X,W)$ is the value that an exponential discounter would obtain from the consumption policy of the sophisticated entrepreneur $C^S(X,W)$. Following Harris and Laibson (2013) we derive the IG Bellman equation for the entrepreneur

$$\delta F^S(X,W) = \frac{(\hat{C}^S)^{1-\gamma}-1}{1-\gamma} + F^S_W(X,W)\left(rW + \hat{\pi}^S W\left(\mu_S - r\right) + X - \hat{C}^S - X\left(\hat{i}^S + \theta\frac{(\hat{i}^S)^2}{2}\right)\right)$$
$$+ \frac{1}{2}F^S_{WW}(X,W)\left((\hat{\pi}^S)^2 W^2 \sigma_S^2\right) + F^S_X(X,W)(\mu_X + \hat{i}^S)X + \frac{1}{2}F^S_{XX}(X,W)\sigma_X^2 X^2, \tag{3.1}$$

where

$$\hat{C}^S(X,W) = \left(\frac{1}{\beta F^S_W(X,W)}\right)^{1/\gamma}, \tag{3.2}$$

$$\hat{\pi}^S(X,W) = -\frac{(\mu_S - r)F_W^S(X,W)}{\sigma_S^2 F_{WW}^S(X,W)W}, \tag{3.3}$$

$$\hat{i}^S(X,W) = \frac{1}{\theta}\left(\frac{F_X^S(X,W) - F_W^S(X,W)}{F_W^S(X,W)}\right), \tag{3.4}$$

and subject to boundary conditions $F(0,W) = G(W)$ specified in the Appendix and a growth condition that ensures admissibility as per Definition 1. Equation (3.2) corresponds to the standard first-order condition for consumption except that the investor discounts the marginal benefit of wealth by the additional factor β.[7] This mechanism leads to the standard over-consumption result documented in the present-bias literature. Equations (3.3) corresponds to the usual expression for the optimal portfolio. It states that the fraction of financial wealth invested in the risky asset is proportional to premium of the risky asset and inversely proportional to the volatility of the risky asset and the relative risk-aversion implied by the value function. Finally, equation (3.4) equates the marginal cost of increasing X by an additional unit $(1+\theta i)F_W$ with its marginal cost F_X.

For general preferences and stochastic processes the partial differential equation (3.1) is very challenging to solve. However, the combination of CRRA preferences and GBM for the stochastic process X deliver a scaling property that allows us to reduce the state space to a one dimensional problem with the scaled firm size $x = X/W$ as the single state variable. To that end, we guess and verify that

$$F^S(X,W) = f_S(x)W^{1-\gamma} + \frac{1}{\delta}\frac{W^{1-\gamma}-1}{1-\gamma}. \tag{3.5}$$

Substituting this guess into (3.2)-(3.4) yields

$$C^S(X,W) = \underbrace{\left[\left(\beta\left(-xf_S'(x) - \gamma f_S(x) + f_S(x) + \frac{1}{\delta}\right)\right)^{-1/\gamma}\right]}_{c^S(x)} W = c^S(x)W, \tag{3.6}$$

$$\pi^S(X,W) = \pi^S(x) = \frac{(\mu_S - r)\left(\delta x f_S'(x) + (\gamma-1)\delta f_S(x) - 1\right)}{{\sigma_S}^2\left(-\gamma + \delta x\left(x f_S''(x) + 2\gamma f_S'(x)\right) + (\gamma-1)\gamma\delta f_S(x)\right)}, \tag{3.7}$$

$$i^S(X,W) = i^S(x) = \frac{-\delta(x+1)f_S'(x) + f_S(x)(\delta - \gamma\delta) + 1}{\theta\left(\delta x f_S'(x) + (\gamma-1)\delta f_S(x) - 1\right)}. \tag{3.8}$$

Finally, we substitute (3.6), (3.7), (3.8), and our guess (3.5) into the IG Bellman equation (3.1) to

[7] Harris and Laibson (2001) manipulate this expression to derive the Hyperbolic Euler Relation and show investors behave as if they had an endogenous rate of time preference that depends on their marginal propensity to consume.

obtain the ordinary differential equation (ODE):

$$
\begin{aligned}
0 =& \frac{1}{\gamma-1} + \frac{1}{2}\sigma_X{}^2 x^2 f_S''(x) - x f_S'(x)(-\mu_X + r + x) - \frac{(\beta+\gamma-1)\beta^{-1/\gamma}\left(\frac{-\delta x f_S'(x)+(1-\gamma)\delta f_S(x)+1}{\delta}\right)^{\frac{\gamma-1}{\gamma}}}{\gamma-1} \\
&+ \frac{x\left(\delta f_S'(x) + \delta x f_S'(x) + (\gamma-1)\delta f_S(x) - 1\right)^2}{2\theta\delta\left(-\delta x f_S'(x) + (\gamma-1)(-\delta) f_S(x) + 1\right)} + \frac{(r-\mu_S)^2\left(-\delta x f_S'(x) + (1-\gamma)\delta f_S(x) + 1\right)^2}{2\delta\sigma_S{}^2\left(\gamma - \delta x\left(x f_S''(x) + 2\gamma f_S'(x)\right) + (1-\gamma)\gamma\delta f_S(x)\right)} \\
&+ (1-\gamma) f_S(x)(r+x) - \delta f_S(x) + \frac{r+x}{\delta},
\end{aligned}
\tag{3.9}
$$

which verifies that our guess was correct. Because equation (3.9) is a second-order differential equation we need to specify boundary conditions. The first boundary condition corresponds to the value of $x = 0$ and states that

$$f_S(0) = A^S, \tag{3.10}$$

where A^S is a constant specified in the Appendix. Intuitively, because X follows a controlled GBM process, it follows that $X = 0$ is an absorbing boundary for that state, and therefore 0 is also an absorbing boundary for $x = X/W$ (i.e., if $x_t = 0 \implies x_{t+s} = 0$ for all $s \geq 0$). Hence, once x hits 0 we can simply solve the model as if there was no X (i.e., with W as the only state variable). The constant A^S is easily obtained from solving this simplified model.

The second boundary condition is more involved. We need to ensure the dynamics of W_t implied by the consumption and investment policies satisfy the admissibility constraint that W_t be non-negative for all t. To that end, we use Ito's formula to compute the dynamics of $w_t = W_t/X_t = 1/x_t$ to obtain

$$dw_t = \mu_w(w_t)dt + \sigma_w^X(w_t)dB_t^X + \sigma_w^S(w_t)dB_t^S,$$

where the explicit expressions for $\mu_w(w), \sigma_w^X(w)$, and $\sigma_w^S(w)$ as functions of $f(x), f'(x)$, and $f''(x)$ are provided in the Appendix. Admissibility requires that $w_t \geq 0$, which is equivalent to ensuring that

$$\sigma_w^X(w)\big|_{w=0} = 0 \quad \sigma_w^S(w)\big|_{w=0} = 0 \quad \mu_w(w)\big|_{w=0} > 0. \tag{3.11}$$

These conditions ensure that the stochastic processes w_t (and therefore W_t) never go below 0, because the volatilities will vanish at 0 and the drift will be strictly positive. Finally, we solve ODE (3.9) subject to boundary condition (3.10) at $x = 0$ and boundary condition (3.11) at $x \to \infty$ (i.e., at $w \to 0$). This problem requires a delicate mathematical treatment because of the presence of singularities at $x = 0$ and at $x \to \infty$. We refer the interested reader to the Appendix for details in how to deal with these singularities.

3.2 Solution under naive beliefs

We now turn to the case of naive beliefs. The investor believes (incorrectly) that only her current self displays present-bias, but that her future selves will behave according to standard exponential discounting. We solve the model recursively by considering the behavior of an exponential discounter. To that end, we set $\beta = 1$ in the model of Section 3.1, and denote with the superscript E the value for the exponential discounter $F^E(X, W) = F^S(X, W; \beta = 1)$. The Bellman equation for the naive investor $F^N(X, W)$ is given by

$$\delta F^N(X,W) = \frac{(\hat{C}^N)^{1-\gamma} - 1}{1-\gamma} + F^N_W(X,W)\left(rW + \hat{\pi}^N W\left(\mu_S - r\right) + X - \hat{C}^N - X\left(\hat{i}^N + \theta\frac{(\hat{i}^N)^2}{2}\right)\right)$$
$$+ \frac{1}{2}F^N_{WW}(X,W)\left((\hat{\pi}^N)^2 W^2 \sigma_S^2\right) + F^N_X(X,W)(\mu_X + \hat{i}^N)X + \frac{1}{2}F^N_{XX}(X,W)\sigma_X^2 X^2, \tag{3.12}$$

where

$$\hat{C}^N(X,W) = \left(\frac{1}{\beta F^E_W(X,W)}\right)^{1/\gamma}, \tag{3.13}$$

$$\hat{\pi}^N(X,W) = -\frac{(\mu_S - r)F^E_W(X,W)}{\sigma_S^2 F^E_{WW}(X,W)W}, \tag{3.14}$$

$$\hat{i}^N(X,W) = \frac{1}{\theta}\left(\frac{F^E_X(X,W) - F^E_W(X,W)}{F^E_W(X,W)}\right). \tag{3.15}$$

We note that in contrast to the policies chosen by the sophisticated investor (problem (3.2) - (3.4)), the naive entrepreneur fails to anticipate that her future selves will also display present-bias. As a result, her policies reflect the (incorrect) belief that her future selves will behave as she would like them to. Mathematically, this can be seen by noting that the RHS of equations (3.6) - (3.8) use the value function of the exponential discounter $F^E(X, W)$ when computing consumption, portfolio, and investment policies.

The model can be solved in the same fashion as the model for the sophisticated entrepreneur.

That is, we guess and verify that

$$F^N(X,W) = f_N(x)W^{1-\gamma} + \frac{1}{\delta}\frac{W^{1-\gamma}-1}{1-\gamma}, \tag{3.16}$$

and by solving an associated ODE for $f_N(x)$ with suitable boundary conditions we obtain the policies of the naive investor $c^N(x), \pi^N(x)$, and $i^N(x)$.

4 Results

Our results are organized as follows. We first prove our main result characterizing the welfare implication of sophistication on present-biased entrepreneurs as a function of their relative risk aversion in a simplified model without an illiquid asset. Then, we numerically show our main result holds in the presence of an illiquid asset. Moreover, we characterize the role played by illiquidity and volatility on the relative welfare of the sophisticated and the naive investors. Finally, we study the steady-state distribution implied by our model, and show that from the perspective of a present-bias investor the equity premium puzzle Mehra and Prescott (1985) and the private equity puzzle (Moskowitz and Vissing-Jørgensen, 2002) are two sides of the same coin.

4.1 Analytical results: the case without illiquid asset

We start by recalling that, following most of the literature, our welfare analysis relies on the long-run welfare criterion put forth by O'Donoghue and Rabin (1999, 2001). This criterion assesses the welfare of an investor by the value that an exponential discounter would derive from the policies chosen by such investor. In our notation the welfare of the sophisticated (resp. naive) investor is given by $F^S(X,W)$ (resp. $F^N(X,W)$).

In this section we specialize our results to the case in which the investor cannot accumulate wealth in the non-traded asset (i.e., we assume $X_t = 0$ for all t). In this case, the value for the sophisticated investor (3.5) and associated consumption policy (3.6) simplify to:

$$F^S(0,W) = F^S(W) = A^S W^{1-\gamma} + \frac{1}{\delta}\frac{W^{1-\gamma}-1}{1-\gamma},$$

$$C_S(W) = \underbrace{\left[\left(\beta\left(-\gamma A^S + A^S + \frac{1}{\delta}\right)\right)^{-1/\gamma}\right]}_{c_S} W = c_S W, \tag{4.1}$$

while the value for the naive present biased investor (3.16) and associated consumption policy (3.13) are given by:

$$F^N(0,W) = F^N(W) = A^N W^{1-\gamma} + \frac{1}{\delta}\frac{W^{1-\gamma}-1}{1-\gamma},$$

$$C_N(W) = \underbrace{\left[\left(\beta\left(-\gamma A^E + A^E + \frac{1}{\delta}\right)\right)^{-1/\gamma}\right]}_{c_N} W = c_N W, \tag{4.2}$$

where $A^E = f_E(0)$.

Condition 2. We make the following technical assumption on the model's parameters

$$(\beta+\gamma-1)\left[\delta + \left(r + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right] > 0.$$ [8]

We are now ready to state our main result in this setting in the following two Propositions:

Proposition 3. *Under Condition 2, the present-biased naive investor over-consumes more than the present-biased sophisticated investor if and only if the coefficient of relative risk-aversion is greater than one:*

$$c_N > c_S \iff \gamma > 1.$$

Proposition 4. *Under Condition 2, the welfare of the present-biased naive investor is lower than that of the present-biased sophisticated investor if and only if the coefficient of relative risk-aversion is greater than one:*

$$A^N < A^S \iff \gamma > 1.$$

Panel A of Figure 4.1 illustrates Proposition 3 by depicting $C_N(W)$ and $C_S(W)$ for the case $\gamma = 2 > 1$. Indeed, the the consumption of the naive investor is larger than that of the sophisticated investor. Similarly, Panel B illustrates Proposition 4 by depicting $F_E(W)$ and $F_S(W)$ and showing that the naive investor is worse-off than the sophisticated investor given our long-run welfare criterion. Panels C and D show that the converse is true for the case when $\gamma = 1/2 < 1$.

The intuition for our results is as follows. Because the sophisticated investor is aware that her future selves will also display present-bias, our results can be explained as the interaction of two effects. On the one hand, the sophisticated investor anticipates that her future selves will splurge, thus making saving less appealing, and leading to more over-consumption. We refer to

[8] For the standard calibration of $\gamma \geq 1$, this condition automatically holds for any value of $\beta > 0$ and $\delta > 0$.

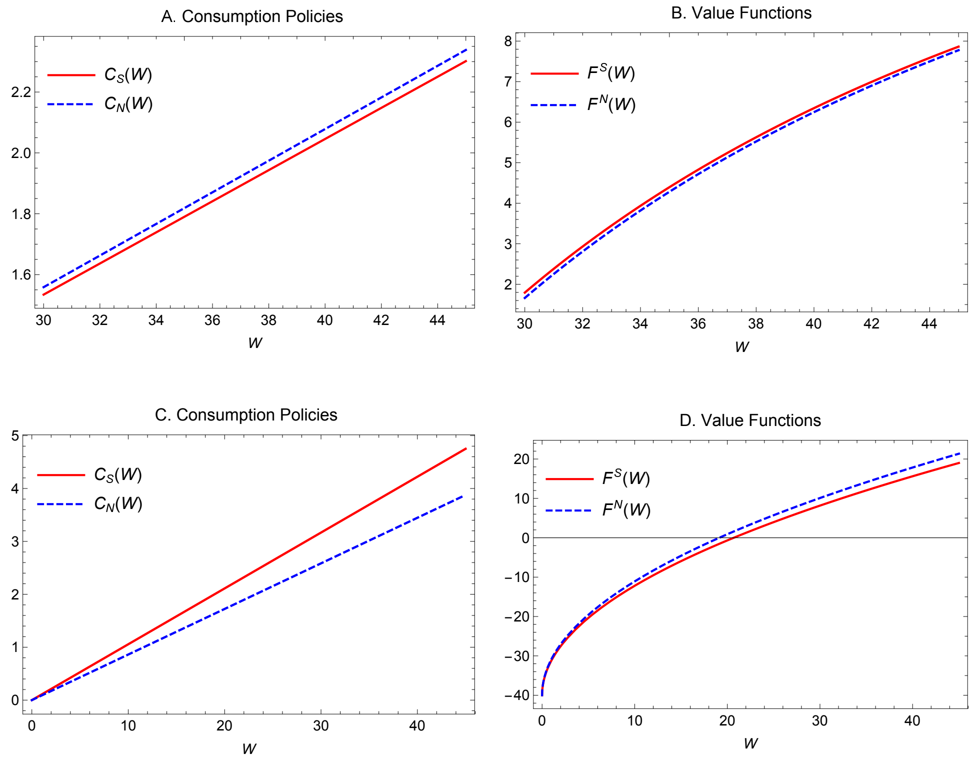

Figure 4.1: Illustration for Propositions 3 and 4. Top (resp. bottom) panels depict consumption and welfare for $\gamma = 2$ (resp. $\gamma = 1/2$). Parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05$ and $\beta = 0.7$.

this mechanism as the splurging effect. On the other hand, the sophisticated investor correctly anticipates that splurging will lead her future selves into low wealth regions (i.e., poverty) where the marginal utility of consumption is very high. Thus, poverty prevention makes saving by the current-self more appealing, thereby leading to less over-consumption. We refer to this force as the precautionary effect. It turns out that when $\gamma > 1$ the precautionary effect dominates and the sophisticated investor consumes more than the naive investor, while when $\gamma < 1$ the splurging effect dominates and the reverse happens. Finally, for logarithmic preferences ($\gamma = 1$) these two effects exactly cancel each other out and the consumption policies of both types of investors coincide.[9]

4.2 Numerical results: overconsumption and welfare

In this section we provide numerical solutions. The annualized portfolio choice parameters follow a standard calibration: interest rate $r = 3\%$, expected return on the risky asset $\mu_S = 6\%$, and

[9]Pollak (1968) showed the equivalence between sophistication and naiveté for log preferences in a setup with similar preferences and without uncertinaty. Marin-Solano and Navas (2010) showed that portfolio rules for the sophisticated and naive investor coincide within the Merton model for logarithmic preferences.

volatility of the risky asset $\sigma_S = 18\%$. We also set the baseline growth rate of private equity $\mu_X = 1\%$, the volatility of private equity $\sigma_X = 10\%$, and the adjustment costs $\theta = 20000$ to match the growth rate and volatility of non-tradable income in Viceira (2001). Finally, we set the long-term subjective discount rate $\delta = 5\%$ and the present-bias parameter $\beta = 0.7$ following Angeletos *et al.* (2001).

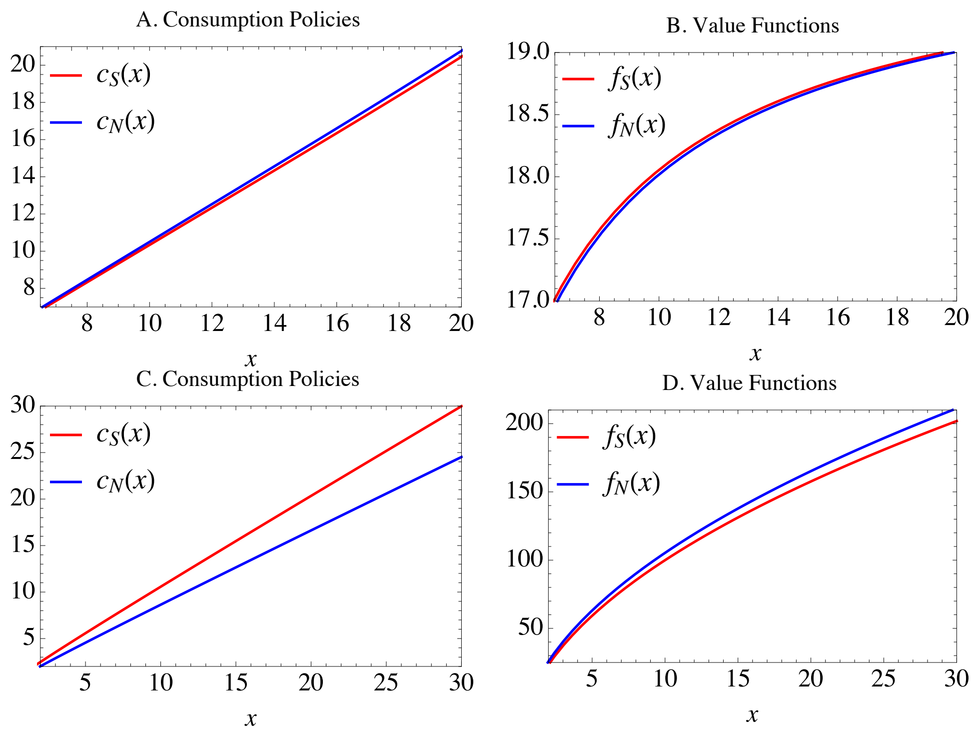

Figure 4.2: Overconsumption and welfare. Top (resp. bottom) panels depict consumption policy and welfare for $\gamma = 2$ (resp. $\gamma = 1/2$). Other parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05, \mu_X = 0.01, \sigma_X = 0.10, \theta = 20000$ and $\beta = 0.7$.

We now return to the full model where the investor has also access to investment in the non-traded (illiquid) asset $X_t > 0$. We start by illustrating numerically that the implications of Propositions 3 and 4 remain true in the presence of illiquid investment technologies. Panel A of Figure 4.2 depicts the consumption policies for the sophisticated investor $c_S(s)$ and the naive investor $c_N(x)$ for $\gamma = 2$. Exactly as in the case without illiquid assets, the precautionary effect dominates the splurging effect inducing the sophisticated investor to save more, and therefore consume less than her naive counterpart. The greater overconsumption of the naive investor makes her worse-off than the sophisticated investor under the long-run welfare criteria, as depicted in Panel B. That is, $f_N(x) \leq f_S(x)$ for all values of x.

By contrast, panel C shows that the consumption policy of the naive investor lies below that of the sophisticated investor when $\gamma = 1/2$. In this case, the splurging effect dominates the precautionary effect inducing the sophisticated investor to save less and therefore consume more than her naive counterpart. As a consequence, the naive investor is better-off than the sophisticated investor. That is, $f_N(x) \geq f_S(x)$, as seen in Panel D.

We now proceed to derive further insights by exploring the role played by illiquidity and volatility on the investor's consumption decision and her associated welfare.

4.2.1 The value of illiquidity

Figure 4.3 depicts comparative statics with respect to the parameter governing the degree of irreversibility in investment θ, which can be interpreted as a proxy for illiquidity. Panel A shows the wedge in consumption policies between the sophisticated and the naive investor $c_N(x) - c_S(x) \geq 0$ when $\gamma = 2$. Panel B depicts the associated welfare gap $f_S(x) - f_N(x) \geq 0$ for this case. Both of these gaps are increasing in the degree of illiquidity θ. Intuitively, the sophisticated investor is aware that her future selves will also display present-bias, and therefore can use the illiquid private equity X as a commitment device to prevent her future selves from splurging. By contrast, the naive investor does not anticipate her future selves to splurge and thus does not perceive any benefit from illiquidity. As a result, the welfare gap $f_S(x) - f_N(x)$ is increasing in θ.

Panels C and D depict the case $\gamma = 1/2$, which corresponds to a instance in which the sophisticated over-consumes more than the naive. Because it is more intuitive to think in terms of positive gaps, panel C depicts the consumption gap $c_S(x) - c_N(x) \geq 0$ and panel D the welfare gap $f_N(x) - f_S(x) \geq 0$. In this case, the welfare gap is decreasing in the degree of illiquidity θ. The same mechanism is present in this case: the sophisticated investor is the only one that stands to benefit from asset illiquidity due to its commitment properties. Because in the case $\gamma < 1$ the sophisticated investor is worse-off than the naive investor, illiquidity allows her to close that gap and bring her welfare closer to that of the naive investor.

We conclude this section by noting that naiveté and high risk-aversion constitute a very problematic combination from a welfare perspective: not only is the naive investor worse-off than her sophisticated counterpart, the naive investor would not benefit from the availability of illiquid investment opportunities.

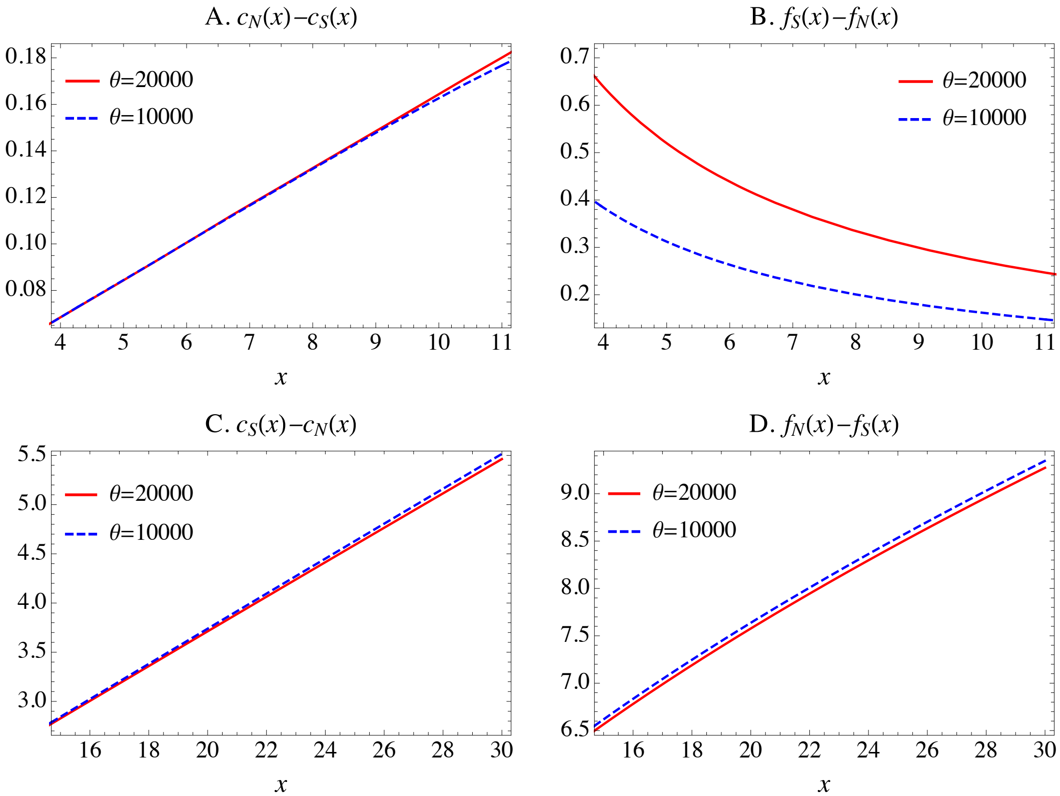

Figure 4.3: The value of illiquidity. Top (resp. bottom) panel depicts the relative over-consumption and welfare gap for $\gamma = 2$ (resp. $\gamma = 1/2$). Other parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05, \mu_X = 0.01, \sigma_X = 0.10, \theta = 20000$ and $\beta = 0.7$.

4.2.2 The role of uncertainty

We now perform comparative statics with respect to the volatility of the two sources of risk in our model, namely the risky stock and the illiquid private equity. We start with comparative statics with respect to σ_S. Figure 4.4 shows that an increase in σ_S always makes the sophisticated investor relatively better-off than her naive counter-part. This is true both when $\gamma = 1/2$ (panel A) and $\gamma = 2$ (panel B). Intuitively, increasing risk-aversion γ and increasing risk σ_S both activate the precautionary motive (i.e., the possibility that her future selves will be in states with very high marginal utility), encouraging savings, and thereby leading to less overconsumption and higher welfare. Thus, sophistication is more valuable in environments with higher uncertainty.

The comparative statics with respect to σ_X are more involved. Figure 4.5 shows that when $\gamma = 1/2$ the sophisticated investor is relatively better off than the naive investor when the risk of the illiquid asset σ_X increases (panel A). However, when $\gamma = 2$, the opposite holds (panel B). Increasing σ_X has two effects on the welfare of the sophisticated investor. On the one hand,

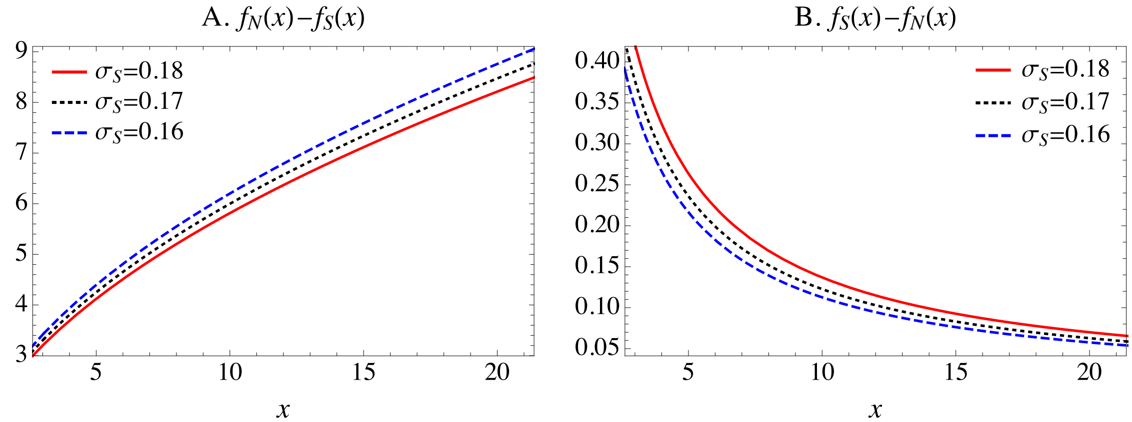

Figure 4.4: Volatility of traded (liquid) asset. Right (resp. left) panel depicts the welfare gap for $\gamma = 2$ (resp. $\gamma = 1/2$). Other parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05, \mu_X = 0.01, \sigma_X = 0.10, \theta = 20000$ and $\beta = 0.7$.

just as in the comparative static with respect to σ_S, increasing risk activates the precautionary motive which uniformly benefits the sophisticated investor. On the other hand, the value of the illiquid asset as a commitment device is undermined the more volatile the return on this asset is. Note that the value of a commitment device arises from allowing the current self to influence the consumption of her future selves; and as the volatility increases, this ability decreases, and therefore the precautionary motive of a (sophisticated) agent. It turns out that when $\gamma = 1/2$ the precautionary motive dominates, but when $\gamma = 2$ the reduction in commitment value dominates.

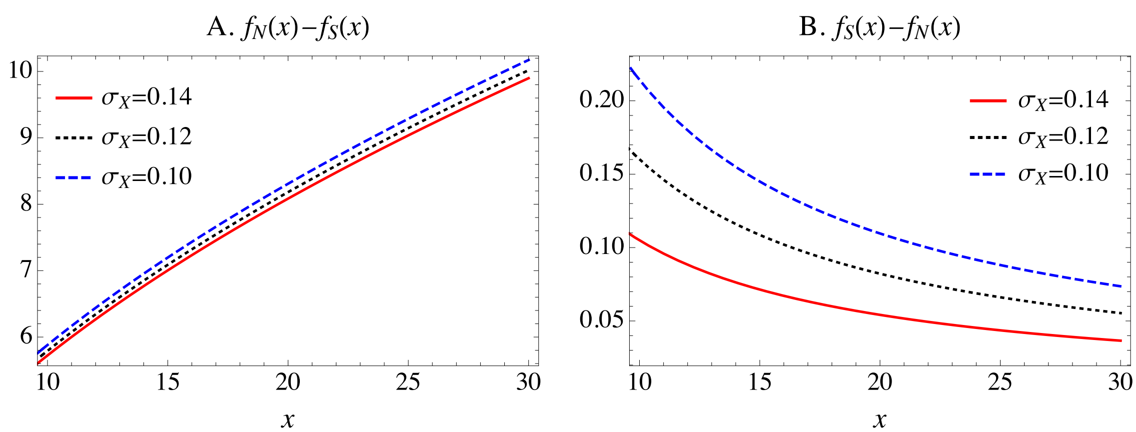

Figure 4.5: Volatility of non-traded (illiquid) asset. Right (resp. left) panel depicts the welfare gap for $\gamma = 2$ (resp. $\gamma = 1/2$). Other parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05, \mu_X = 0.01, \sigma_X = 0.10, \theta = 20000$ and $\beta = 0.7$.

We emphasize that even though we only show numerical results for the cases $\gamma = 1/2$ and

$\gamma = 2$, our numerical explorations suggest that our results are much more general, and generically hold whenever $\gamma < 1$ and $\gamma > 1$, respectively.

4.2.3 Investment in private equity vs risky stock

In this section we study the long-run consequences of present-bias and sophistication attitudes on the long-run distribution of wealth, asset accumulation, and welfare gaps. We first start with a short technical note on the way in which we use our continuous time model to compute the system's ergodic distribution. We then proceed to analyze the implications of model parameters on this distribution for our various cases of interest.

Ergodic distribution

We can easily compute the ergodic distribution for each of our models. Denote the ergodic distribution (or sometimes also called steady-state distribution) by $\phi(x)$ and recall that it is interpreted as the fraction of investors that have a scaled firm size of $x = X/W$ at any given time.

To compute the ergodic distribution we use Ito's formula to obtain the dynamics of $x_t = X_t/W_t$

$$dx_t = \mu_x(x_t)dt + \sigma_x^X(x_t)dB_t^X + \sigma_x^S(x_t)dB_t^S, \tag{4.3}$$

where the functions $\mu_x(x), \sigma_x(x)$, and $\sigma^S(x_t)$ are specified in the Appendix. We note that the specific dynamics of x_t (and therefore functions $\mu_x(x), \sigma_x(x)$, and $\sigma^S(x_t)$) depend on whether the investor holds sophisticated versus naive beliefs.

Endowed with dynamics (4.3) we know that the ergodic distribution $\phi(x)$ satisfies the Fokker-Plank equation

$$0 = -\frac{\partial}{\partial x}[\mu_x(x)\phi(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[(\sigma_x^X(x_t))^2\phi(x) + (\sigma_x^S(x_t))^2\phi(x)],$$

subject to boundary conditions $\phi(0) = \phi(\infty) = 0$ and $\int_0^\infty \phi(x)dx = 1$.

Relation to asset pricing puzzles

The equity premium puzzle (Mehra and Prescott, 1985) essentially states that observed historically large equity returns cannot be easily reconciled within a representative agent economy with standard preferences. Such a model predicts that the demand for the risky asset is too large to induce the underpricing needed for the risky asset to generate the observed high returns. By contrast, the private equity puzzle (Moskowitz and Vissing-Jørgensen, 2002) states that returns in non-publicly traded equity are no better than those of publicly traded equities. Since entrepreneurial investment is highly concentrated, it is puzzling why investors would display such large demand for this asset class. To summarize, a standard model predicts that demand for publicly traded equities is too large compared to that observed in reality, while demand for private equities is too small.

A model with present bias investors can simultaneously shade light on these two facts. We show that present-bias leads the investor to allocate a larger fraction of her wealth to the private equity (because of its commitment value) and a smaller fraction of her wealth to publicly traded equities.

To do so, we define the total value of the investor's wealth TW_t as the sum of her financial wealth W_t plus the wealth from her private equity $\frac{X_t}{r-\mu_X}$, where for simplicity we assume that the dollar value from private equity is computed under the assumption of no investment:

$$TW_t = W_t + \frac{X_t}{r-\mu_X} = W_t\Big(1 + \frac{x_t}{r-\mu_X}\Big).$$

Thus, the fraction of wealth the investor keeps in the traded and non-traded assets are respectively given by

$$\frac{W_t}{W_t(1+\frac{x_t}{r-\mu_X})} = \frac{r-\mu_X}{r-\mu_X+x_t},$$

$$\frac{\frac{X_t}{r-\mu_X}}{W_t(1+\frac{x_t}{r-\mu_X})} = \frac{x_t}{r-\mu_X+x_t}.$$

Finally, denoting the long-run fraction of wealth the investor allocates to the traded (resp. non-traded) asset by α (resp. $1-\alpha$), we proceed to compute these quantities relying on the steady-state distributions computed in the previous section. That is,

$$\alpha = \int_0^\infty \frac{r-\mu_X}{r-\mu_X+x}\phi(x)dx.$$

$$1-\alpha = \int_0^\infty \frac{x}{r-\mu_X+x}\phi(x)dx.$$

Figure 4.6 depicts the fraction of wealth allocated to the traded asset as a function of β for both naive and sophisticated investors for three different values of γ. We notice that the fraction of wealth allocated to the (liquid) traded asset is increasing in β (i.e., the fraction of wealth allocated to the non-traded (illiquid) asset is decreasing in β). As a result, our model can simultaneously explain why present-biased investors would have a stronger preference for private equity versus publicly traded stocks. Thus, the equity premium and the private equity puzzles, from the lense of a present-biased investor, are two sides of the same coin. Importantly, this result is independent on whether the agent is sophisticated or naive, and whether her risk-aversion is large or small.

We conclude this section by noting that, just as what happens to the consumption policies, the investment policies of the naive and sophisticated investor coincide for the logarithmic case of $\gamma = 1$(panel B), but differ for the case general case when $\gamma < 1$ (panel A) and $\gamma > 1$ (panel C). That is, the fraction of wealth allocated to the traded-asset α inherits the same properties as the consumption policy $c(x_t)$ of present-bias agents regarding the relationship between sophistication and risk-aversion.

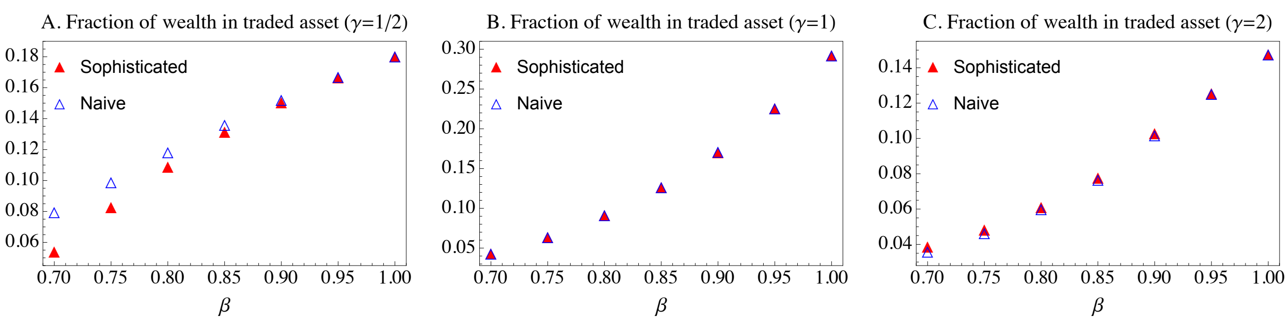

Figure 4.6: Fraction of wealth allocated to the (liquid) traded asset. Baseline parameter values are $r = 0.03$, $\mu_S = 0.06$, $\sigma_S = 0.18$, $\delta = 0.05$, $\mu_X = 0.01$, $\sigma_X = 0.10$, $\theta = 10000$ and $\beta = 0.7$.

5 Conclusion

We analyze a situation in which a risk-averse present-biased investor dynamically chooses her consumption and investment decision. The investment decision consists of allocating funds between liquid traded assets (a risk-free bond and a risky stock) and an illiquid non-traded asset (private equity). In our setting, the present-biased investor can be either naive or sophisticated with respect to her present bias. We characterize the implications of present bias and degree of sophistication on the investor's decisions and her long-run utility.

Our main finding states that a sophisticated investor over-consumes more than her naive counter-part if and only if her coefficient of relative risk-aversion is smaller than one. As a result, sophistication is welfare enhancing from a long-run utility perspective when the investor is sufficiently risk-averse. Otherwise, the investor would be better-off being naive about her future present bias. We study the role of illiquidity and volatility on the investor's welfare and show that the presence of an illiquid asset disproportionally benefits the sophisticated investor, due to the commitment value of illiquidity. However, such commitment value is depleted as the illiquid asset becomes more volatile.

These results raise several interesting questions for future research. For example, how do sophistication and risk-aversion interact in a real option setting similar to that of Grenadier and Wang (2007)? In a principal-agent context, what are the features of an optimal contract when the agent displays present-bias? How do these features depend on whether the agent is sophisticated or naive, and her degree of risk-aversion? We leave these and other questions for future research.

6 Appendix

Preliminiaries for proofs of Propositions 3 and 4:

Setting $x = 0$ into the corresponding ODEs for $f_E(x), f_S(x)$ and $f_N(x)$, and denoting by $A^E = f_E(0), A^S = f_S(0)$ and $A^N = f_N(0)$ we obtain:

$$\frac{r}{\delta}+\frac{1}{\gamma-1}-A^E(\gamma-1)r-A^E\delta+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)-\frac{\gamma}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)^{\frac{\gamma-1}{\gamma}}=0 \quad (6.1)$$

$$\frac{r}{\delta}+\frac{1}{\gamma-1}-A^S(\gamma-1)r-A^S\delta+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)-\frac{(\beta+\gamma-1)\beta^{-1/\gamma}}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)^{\frac{\gamma-1}{\gamma}}=0 \quad (6.2)$$

$$\frac{r}{\delta} + \frac{1}{\gamma-1} - A^N(\gamma-1)r - A^N\delta + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\left(\frac{1}{\delta}-(\gamma-1)A^N\right) \quad (6.3)$$

$$- \beta^{-1/\gamma}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)^{\frac{-1}{\gamma}}\left[\frac{\beta}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)+\left(\frac{1}{\delta}-(\gamma-1)A^N\right)\right]=0 \quad (6.4)$$

We make the following claims:

1. $A^E > A^S$ and $A^E > A^N$ for $\gamma > 0$.
2. $A^S > A^N$ for $\gamma > 1$.
3. $A^S < A^N$ for $\gamma \in (0,1)$.

Proof of $A^E > A^S$:

Equation (6.2) can be written as

$$r\left(\frac{1}{\delta}-(\gamma-1)A^S\right) + \frac{\delta}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)$$

$$- \frac{(\beta+\gamma-1)\beta^{-1/\gamma}}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)^{\frac{\gamma-1}{\gamma}}=0 \quad (6.5)$$

We consider $\left(\frac{1}{\delta}-(\gamma-1)A^S\right) \neq 0$ otherwise $A^S = \frac{1}{\delta(\gamma-1)} \to \infty$ as $\gamma \to 1$. Dividing (6.2) by $\left(\frac{1}{\delta}-(\gamma-1)A^S\right)$, we get

$$\frac{(\beta+\gamma-1)\beta^{-1/\gamma}}{\gamma-1}\left(\frac{1}{\delta}-(\gamma-1)A^S\right)^{\frac{-1}{\gamma}} = r+\frac{\delta}{\gamma-1}+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}$$

$$\Rightarrow \quad \left(\frac{1}{\delta}-(\gamma-1)A^S\right)^{\frac{-1}{\gamma}} = \frac{\beta^{1/\gamma}}{(\beta+\gamma-1)}\left[\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right] \tag{6.6}$$

This gives

$$\begin{aligned}
A^S &= \frac{1}{\gamma-1}\left[\frac{1}{\delta}-\frac{(\beta+\gamma-1)^\gamma}{\beta}\left\{\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right\}^{-\gamma}\right] \\
&= \frac{1}{\gamma-1}\left[\frac{1}{\delta}-\frac{(\beta+[1+\beta\ln(\beta)](\gamma-1)}{\beta}\left\{\frac{1}{\delta}+\frac{-r-\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}-\delta\ln(\delta)}{\delta^2}(\gamma-1)\right\}\right]+O(\gamma-1) \\
&= \frac{r}{\delta^2}+\frac{(r-\mu_S)^2}{2\delta^2\sigma_S^2}+\frac{\ln(\delta)}{\delta}-\frac{1}{\beta\delta}-\frac{\ln(\beta)}{\delta} \qquad \text{as} \qquad \gamma\to 1
\end{aligned}$$

This proves that A^S is finite as $\gamma \to 1$.

Denote

$$Z=\left(\frac{1}{\delta}-(\gamma-1)A^S\right)^{\frac{-1}{\gamma}}$$

Then (6.6) gives

$$Z=\frac{\beta^{1/\gamma}}{(\beta+\gamma-1)}\left[\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right]$$

and then

$$\begin{aligned}
\frac{\partial Z}{\partial\beta} &= \frac{\beta^{1/\gamma}}{(\beta+\gamma-1)^2}\left(\frac{\beta+\gamma-1}{\gamma\beta}-1\right)\left[\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right] \\
&= \frac{Z}{\beta+\gamma-1)}\left(\frac{1}{\beta}-1\right)\left(1-\frac{1}{\gamma}\right)
\end{aligned}$$

Condition

$$(\beta+\gamma-1)\left[\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right]>0$$

imply that

$$(\gamma-1)\frac{\partial Z}{\partial\beta}>0.$$

Case 1 ($\gamma > 1$):

$$\Rightarrow \quad Z \text{ increases with } \beta$$

$$\Rightarrow \quad Z^{-\gamma} = \frac{1}{\delta} - (\gamma - 1)A^S \text{ decreases with } \beta$$

$$\Rightarrow \quad A^S \text{ increases with } \beta$$

Case 2 ($\gamma < 1$)**:**

$$\Rightarrow \quad Z \text{ decreases with } \beta$$

$$\Rightarrow \quad Z^{-\gamma} = \frac{1}{\delta} - (\gamma - 1)A^S \text{ increases with } \beta$$

$$\Rightarrow \quad A^S \text{ increases with } \beta$$

In both cases, A^S increases with β.

Since A^E corresponds to A^S with $\beta = 1$ (largest β), $A^E > A^S$.

Therefore,

$$A^E > A^S \tag{6.7}$$

Proof of $A^E > A^N$:

(6.1) gives

$$E = \left(\frac{1}{\delta} - (\gamma - 1)A^E\right)^{\frac{-1}{\gamma}} = \frac{1}{\gamma}\left[\delta + \left(r + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma - 1)\right] \tag{6.8}$$

and (6.3) gives

$$\begin{aligned}\left(\frac{1}{\delta} - (\gamma - 1)A^N\right) & \left[\frac{\delta}{\gamma - 1} + r + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2} - \beta^{-1/\gamma}\left(\frac{1}{\delta} - (\gamma - 1)A^E\right)^{\frac{-1}{\gamma}}\right] \\ &= \beta^{-1/\gamma}\left(\frac{1}{\delta} - (\gamma - 1)A^E\right)^{\frac{-1}{\gamma}+1}\frac{\beta}{\gamma - 1}\end{aligned}$$

This implies

$$\begin{aligned}
&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)\left\{\frac{1}{\gamma-1}\left[\delta+\left(r+\frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right]-\beta^{-1/\gamma}E\right\}=\frac{\beta^{1-1/\gamma}}{\gamma-1}E^{1-\gamma}\\
\Rightarrow\quad&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)\left[\frac{\gamma E}{\gamma-1}-\beta^{-1/\gamma}E\right]=\frac{\beta^{1-1/\gamma}}{\gamma-1}E^{1-\gamma}\\
\Rightarrow\quad&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)\left[\frac{\gamma}{\gamma-1}-\beta^{-1/\gamma}\right]=\frac{\beta^{1-1/\gamma}}{\gamma-1}E^{-\gamma}\\
\Rightarrow\quad&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)=\frac{\beta^{1-1/\gamma}}{\left(\gamma-(\gamma-1)\beta^{-1/\gamma}\right)}E^{-\gamma}=\frac{\beta}{\left(\gamma\beta^{1/\gamma}-\gamma+1\right)}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)
\end{aligned}$$

Define

$$F(\beta)=\frac{\beta}{\left(\gamma\beta^{1/\gamma}-\gamma+1\right)}-1.$$

Then we get

$$F'(\beta)=\frac{(\gamma-1)(\beta^{\frac{1}{\gamma}}-1)}{\left(1+(\beta^{\frac{1}{\gamma}}-1)\gamma\right)^2}$$

We observe that $F(1)=0$.

Then, $\gamma>1$ implies

$$\begin{aligned}
&F'(\beta)<0\\
\Rightarrow\quad&F(\beta)>0 \text{ for } \beta<1\\
\Rightarrow\quad&\frac{\beta}{\left(\gamma\beta^{1/\gamma}-\gamma+1\right)}>1\\
\Rightarrow\quad&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)=\frac{\beta}{\left(\gamma\beta^{1/\gamma}-\gamma+1\right)}\left(\frac{1}{\delta}-(\gamma-1)A^E\right)>\left(\frac{1}{\delta}-(\gamma-1)A^E\right)\\
\Rightarrow\quad&\left(\frac{1}{\delta}-(\gamma-1)A^N\right)>\left(\frac{1}{\delta}-(\gamma-1)A^E\right)\\
\Rightarrow\quad&A^E>A^N.
\end{aligned}$$

and $\gamma < 1$ implies

$$\begin{aligned}
& F'(\beta) > 0 \\
\Rightarrow\ & F(\beta) < 0 \text{ for } \beta < 1 \\
\Rightarrow\ & \frac{\beta}{(\gamma\beta^{1/\gamma} - \gamma + 1)} < 1 \\
\Rightarrow\ & \left(\frac{1}{\delta} - (\gamma-1)A^N\right) = \frac{\beta}{(\gamma\beta^{1/\gamma} - \gamma + 1)}\left(\frac{1}{\delta} - (\gamma-1)A^E\right) < \left(\frac{1}{\delta} - (\gamma-1)A^E\right) \\
\Rightarrow\ & \left(\frac{1}{\delta} - (\gamma-1)A^N\right) < \left(\frac{1}{\delta} - (\gamma-1)A^E\right) \\
\Rightarrow\ & A^E > A^N.
\end{aligned}$$

In both cases

$$A^E > A^N.$$

Proof of Proposition 3

We have from (6.7),

$$A^E > A^S$$

This gives

$$\begin{aligned}
\gamma > 1 \iff\ & (\gamma-1)A^E > (\gamma-1)A^S \\
\iff\ & \frac{1}{\delta} - (\gamma-1)A^E < \frac{1}{\delta} - (\gamma-1)A^S \\
\iff\ & \left[\beta\left(\frac{1}{\delta} - (\gamma-1)A^E\right)\right]^{-1/\gamma} > \left[\beta\left(\frac{1}{\delta} - (\gamma-1)A^S\right)\right]^{-1/\gamma} \\
\iff\ & c_N > c_S
\end{aligned}$$

Proof of Proposition 4

From (6.2) and (6.3), we get

$$\left(\frac{1}{\delta} - (\gamma-1)A^S\right) = \frac{(\beta+\gamma-1)^\gamma}{\beta}\left[\delta + \left(r + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right]^{-\gamma} \tag{6.9}$$

$$\left(\frac{1}{\delta} - (\gamma-1)A^N\right) = \frac{\beta}{(\gamma\beta^{1/\gamma} - \gamma + 1)}\left[\frac{1}{\gamma}\left\{\delta + \left(r + \frac{(r-\mu_S)^2}{2\gamma\sigma_S^2}\right)(\gamma-1)\right\}\right]^{-\gamma} \tag{6.10}$$

Claim:

$$A^N < A^S \left(A^N > A^S\right) \Longleftrightarrow \gamma > 1(\gamma < 1)$$

This is equivalent to

$$\frac{1}{\delta} - (\gamma - 1)A^S < \frac{1}{\delta} - (\gamma - 1)A^N$$

i.e.

$$\frac{(\beta + \gamma - 1)^\gamma}{\beta} < \frac{\beta}{\gamma\beta^{\frac{1}{\gamma}} - \gamma + 1}\gamma^\gamma$$

$$\frac{(\beta + \gamma - 1)^\gamma}{\gamma^\gamma} < \frac{\beta^2}{\gamma\beta^{\frac{1}{\gamma}} - \gamma + 1} \tag{6.11}$$

$$\beta \geq \frac{1}{2} > \frac{1}{e} > \left(1 - \frac{1}{\gamma}\right)^\gamma$$

$$\Rightarrow \quad \beta^{\frac{1}{\gamma}} > 1 - \frac{1}{\gamma} \quad \Rightarrow \quad \gamma\beta^{\frac{1}{\gamma}} > \gamma - 1 \quad \Rightarrow \quad \gamma\beta^{\frac{1}{\gamma}} - \gamma + 1 > 0$$

We prove (6.11) for

$$1 > \beta \geq \frac{1}{2}, \gamma > 0.$$

Consider the function

$$F(\beta, \gamma) = \beta^2 \left(\frac{\beta}{\gamma} + 1 - \frac{1}{\gamma}\right)^{-\gamma} - \left(\gamma\beta^{\frac{1}{2\gamma}} - \gamma + 1\right).$$

Then we need to show that $F(\beta, \gamma) > 0$ for $\gamma \neq 1, \quad \beta < 1.$

Taylor's expansion with a fixed γ near $\beta = 1$ becomes

$$F(\beta, \gamma) = F(1, \gamma) + \frac{\partial F}{\partial \beta}(1, \gamma)(\beta - 1) + \frac{1}{2}\frac{\partial^2 F}{\partial \beta^2}(1, \gamma)(\beta - 1)^2 + \sum_{k=3}^{\infty} \frac{1}{k!}\frac{\partial^k F}{\partial \beta^k}(1, \gamma)(\beta - 1)^k \tag{6.12}$$

By direct calculation the first three terms become zero and we get

$$F(\beta, \gamma) = \sum_{k=3}^{\infty} \frac{1}{k!}\frac{\partial^k F}{\partial \beta^k}(1, \gamma)(\beta - 1)^k \tag{6.13}$$

Before proceeding further, we make the following three observations:

1.
$$\frac{\partial^k}{\partial\beta^k}\left(\frac{\beta}{\gamma}+1-\frac{1}{\gamma}\right)^{-\gamma}\Bigg|_{\beta=1}=(-1)^k\frac{(1+\gamma)(2+\gamma)\cdots(k-1+\gamma)}{\gamma^{k-1}} \tag{6.14}$$

and for $k\geq 4$, we have :

2.
$$\begin{aligned}
&\frac{\partial^k}{\partial\beta^k}\left[\beta^2\left(\frac{\beta}{\gamma}+1-\frac{1}{\gamma}\right)^{-\gamma}\right]\Bigg|_{\beta=1}\\
=\;&\frac{\partial^k}{\partial\beta^k}\left(\frac{\beta}{\gamma}+1-\frac{1}{\gamma}\right)^{-\gamma}\Bigg|_{\beta=1}+2k\frac{\partial^{k-1}}{\partial\beta^{k-1}}\left(\frac{\beta}{\gamma}+1-\frac{1}{\gamma}\right)^{-\gamma}\Bigg|_{\beta=1}\\
&+k(k-1)\frac{\partial^{k-2}}{\partial\beta^{k-2}}\left(\frac{\beta}{\gamma}+1-\frac{1}{\gamma}\right)^{-\gamma}\Bigg|_{\beta=1}\\
=\;&(-1)^k\frac{(1+\gamma)(2+\gamma)\cdots(k-3+\gamma)}{\gamma^{k-1}}[(k-2+\gamma)(k-1+\gamma)-2k(k-2+\gamma)\gamma+k(k-1)\gamma^2]
\end{aligned}$$

where

$$\begin{aligned}
&(k-2+\gamma)(k-1+\gamma)-2k(k-2+\gamma)\gamma+k(k-1)\gamma^2\\
=\;&\gamma^2(k^2-3k+1)-\gamma(2k^2-6k+3)+k^2-3k+2\\
=\;&(\gamma-1)[(k^2-3k+1)\gamma-(k^2-3k+2)]
\end{aligned}$$

3.
$$\frac{\partial^k}{\partial\beta^k}\left[-\left(\gamma\beta^{\frac{1}{\gamma}}-\gamma+1\right)\right]\Bigg|_{\beta=1}=\frac{(\gamma-1)(2\gamma-1)\cdots((k-1)\gamma-1)}{\gamma^{k-1}}(-1)^k \tag{6.15}$$

Therefore,

$$\begin{aligned}
F(\beta,\gamma)\;=\;&\frac{1}{3!}\frac{\partial^3F}{\partial\beta^3}(1,\gamma)(\beta-1)^3\\
&+\sum_{k=4}^{\infty}\left[(1+\gamma)(2+\gamma)\cdots(k-3+\gamma)(\gamma-1)\left\{\left(k^2-3k+1\right)\gamma-\left(k^2-3k+2\right)\right\}\right.\\
&\left.+(\gamma-1)(2\gamma-1)\cdots((k-1)\gamma-1)\right]\frac{(-1)^k}{k!\gamma^{k-1}}(\beta-1)^k
\end{aligned}$$

where by direct calculation we obtain that

$$\frac{\partial^3F}{\partial\beta^3}(1,\gamma)=-3\frac{(\gamma-1)^2}{\gamma^2}$$

Since $\beta < 1$, in order to show that $F(\beta, \gamma) > 0$, it suffices to show that for each $k \geq 4$,

$$(\gamma - 1)\left[(1+\gamma)(2+\gamma)\cdots(k-3+\gamma)(\gamma-1)\left\{\left(k^2-3k+1\right)\gamma - \left(k^2-3k+2\right)\right\} + (2\gamma-1)\cdots((k-1)\gamma-1)\right] > 0 \tag{6.16}$$

1. Case $\gamma > 1$:

   Let us introduce the variable $g = \gamma - 1$. Then (6.16) is equivalent to

   $$(2+g)(3+g)\cdots(k-2+g)\left[\left(k^2-3k+1\right)g - 1\right] + (1+2g)(2+3g)\cdots(k-2+(k-1)g) > 0,$$

   where $g > 0$. But the left hand side is greater than

   $$(2+3g)(3+4g)\cdots(k-2+(k-1)g) - (2+g)(3+g)\cdots(k-2+g),$$

   which is positive.

   This completes the proof for this case.

2. Case $\gamma < 1$:

   We introduce the variable $\hat{\gamma} = \frac{1}{\gamma}$ and rewrite (6.16) as

   $$(1 - \hat{\gamma})\left[(\hat{\gamma}+1)(2\hat{\gamma}+1)\ldots((k-3)\hat{\gamma}+1)\left\{\left(k^2-3k+1\right) - \left(k^2-3k+2\right)\hat{\gamma}\right\} + (2-\hat{\gamma})(3-\hat{\gamma})\cdots(k-1-\hat{\gamma})\right] > 0,$$

   where $\hat{\gamma} > 1$. Introducing the variable $\hat{g} = \hat{\gamma} - 1 > 0$, we get

   $$(2+\hat{g})(3+2\hat{g})\cdots(k-2+(k-3)\hat{g})\left(-1-\left(k^2-3k+2\right)\hat{g}\right) + (1-\hat{g})(2-\hat{g})\cdots(k-2-\hat{g}) < 0$$

   OR

   $$(2+\hat{g})(3+2\hat{g})\cdots(k-2+(k-3)\hat{g})\left(1+\left(k^2-3k+2\right)\hat{g}\right) - (1-\hat{g})(2-\hat{g})\cdots(k-2-\hat{g}) > 0$$

But

$$\begin{aligned}
2+\hat{g} &> |2-\hat{g}| \\
3+2\hat{g} &> |3-\hat{g}| \\
k-2+(k-3)\hat{g} &> |k-2-\hat{g}| \\
1+\left(k^2-3k+2\right)\hat{g} &> |1-\hat{g}|
\end{aligned}$$

Hence the inequality is true.

In both cases the inequality (6.16) is true. This proves that (6.11) is true and hence our claim is true.

Computations for section 4.2.3:

In order to observe the dynamics of $x_t = \frac{X_t}{W_t}$, we start with the dynamics of X_t and W_t:

$$dX_t = (\mu_X + i_t)X_t dt + \sigma_X X_t dB_t^X \tag{6.17}$$

$$dW_t = [r + \pi_t(\mu_S - r)]\, W_t dt + (X_t - C_t - X_t g(i_t))dt + \pi_t \sigma_S W_t dB_t^S \tag{6.18}$$

Ito's Lemma for $f(X, W)$ gives

$$df = f_X dX + f_W dW + \frac{1}{2} f_{XX} dX^2 + \frac{1}{2} f_{WW} dW^2 + f_{XW} dX dW.$$

Using this Lemma for $f = X_t/W_t$, we get

$$\begin{aligned}
d\left(\frac{X_t}{W_t}\right) &= \frac{1}{W_t} dX_t - \frac{X_t}{W_t^2} dW_t + \frac{X_t}{W_t^3} dW_t^2 - \frac{1}{W_t^2} dX_t dW_t \\
&= \frac{X_t}{W_t}\left[\frac{dX_t}{X_t} - \frac{dW_t}{W_t} + \frac{dW_t^2}{W_t^2} - \frac{dX_t dW_t}{X_t W_t}\right] \\
&= \frac{X_t}{W_t}\Bigg[(\mu_X + i)dt + \sigma_X dB_t^X - (r + \pi_t(\mu_S - r))dt \\
&\qquad -\pi_t \sigma_S dB_t^S - \left(\frac{X_t - C_t - X_t g(i_t)}{W_t}\right) dt + \pi_t^2 \sigma_S^2 dt\Bigg]
\end{aligned} \tag{6.19}$$

because

$$
\begin{aligned}
\frac{dX_t}{X_t} &= (\mu_X + i_t)dt + \sigma_X dB_t^X \\
\frac{dW_t}{W_t} &= (r + \pi_t(\mu_S - r))dt + \left(\frac{X_t - C_t - X_t g(i_t)}{W_t}\right) dt + \pi_t \sigma_S dB_t^S \\
\frac{dW_t^2}{W_t^2} &= \left[(r + \pi_t(\mu_S - r))dt + \left(\frac{X_t - C_t - X_t g(i_t)}{W_t}\right) dt + \pi_t \sigma_S dB_t^S\right]^2 \\
&= \pi_t^2 \sigma_S^2 dt, \\
&\qquad \text{as } \ (dt)^2 \to 0, \ \ dB_t^S dt \to 0 \ \text{ and } \ (dB_t^S)^2 \to dt. \\
\frac{dX_t dW_t}{X_t W_t} &= \left[\frac{dX_t}{X_t}\right]\left[\frac{dW_t}{W_t}\right] \\
&= \left[\mu_X dt + \sigma_X dB_t^X\right]\left[(r + \pi_t(\mu_S - r))dt + \left(\frac{X_t - C_t}{W_t}\right) dt + \pi_t \sigma_S dB_t^S\right] \\
&= 0 + 0 + 0 + \pi \sigma_X \sigma_S dB_t^X dB_t^S, \\
&\qquad \text{as } \ (dt)^2 \to 0, \ \ dB_t^S dt \to 0 \ \text{ and } \ (dB_t^S)^2 \to dt. \\
&= \pi_t \rho \sigma_X \sigma_S dt, \\
&\qquad \text{as } \ dB_t^X dB_t^S \to \rho dt, \ \ \rho = \text{correlation coefficient between } \ B_t^X \ \text{ and } \ B_t^S. \\
&= 0 \quad \text{as} \quad \rho = 0.
\end{aligned}
$$

Then (6.19) implies

$$
\begin{aligned}
dx_t &= x_t\Big[(\mu_X + i_t)dt + \sigma_X dB_t^X - (r + \pi_t(\mu_S - r))dt \\
&\qquad - \pi_t \sigma_S dB_t^S - (x_t - c(x_t) - x_t g(i_t))\, dt + \pi_t^2 \sigma_S^2 dt\Big] \\
&= x_t\Big[\mu_X + i_t - r - \pi_t(\mu_S - r) - x_t + c(x_t) + x_t g(i_t) + \pi_t^2 \sigma_S^2\Big] dt + \sigma_X x_t dB_t^X - \pi_t \sigma_S x_t dB_t^S
\end{aligned}
\tag{6.20}
$$

Equation (6.20) can be expressed as

$$
dx_t = \mu_x(x_t)dt + \sigma_x^X(x_t)dB_t^X + \sigma_x^S(x_t)dB_t^S \tag{6.21}
$$

where

$$
\mu_x(x) = x\Big[\mu_X + i - r - \pi(x)(\mu_S - r) - x + c(x) + xg(i(x)) + \pi(x)^2 \sigma_S^2\Big],
$$

$$
\sigma_x^X(x) = x\sigma_X \ \text{ and } \ \sigma_x^S(x) = -x\sigma_S \pi(x).
$$

Sophisticated case:

$$
\begin{aligned}
\mu_x(x) &= x\Big[\mu_X + i - r - \pi^S(x)(\mu_S - r) - x + c^S(x) + xg(i^S(x)) + (\pi^S(x))^2\sigma_S^2\Big] \\
\sigma_x^X(x) &= x\sigma_X \\
\sigma_x^S(x) &= -x\sigma_S\pi^S(x)
\end{aligned}
$$

where $c^S(x), \pi^S(x)$ and $i^S(x)$ are given in (3.6), (3.7) and (3.8) respectively..

Naive case:

$$
\begin{aligned}
\mu_x(x) &= x\Big[\mu_X + i - r - \pi^N(x)(\mu_S - r) - x + c^N(x) + xg(i^N(x)) + (\pi^N(x))^2\sigma_S^2\Big] \\
\sigma_x^X(x) &= x\sigma_X \\
\sigma_x^S(x) &= -x\sigma_S\pi^N(x)
\end{aligned}
$$

where

$$c^N(x) = \left[\left(\beta\left(-xf_E'(x) - \gamma f_E(x) + f_E(x) + \frac{1}{\delta}\right)\right)^{-1/\gamma}\right] \tag{6.22}$$

$$\pi^N(x) = \frac{(\mu_S - r)\left(\delta x f_E'(x) + (\gamma-1)\delta f_E(x) - 1\right)}{\sigma_S{}^2\left(-\gamma + \delta x\left(x f_E''(x) + 2\gamma f_E'(x)\right) + (\gamma-1)\gamma\delta f_E(x)\right)} \tag{6.23}$$

$$i^N(x) = \frac{-\delta(x+1)f_E'(x) + f_E(x)(\delta - \gamma\delta) + 1}{\theta\left(\delta x f_E'(x) + (\gamma-1)\delta f_E(x) - 1\right)} \tag{6.24}$$

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